1 Loan Payments, Credit Cards, and Mortgages

1.1 Loan Basics

Definition 1.1: For any loan, the principal is the amount of money owed at any particular time. Interest is charged on the loan principal. To pay the loan, you must gradually pay down the principal.
Definition 1.2: The loan term is the time you have to pay back the loan in full

How many people think, when you have a loan with a 10% interest rate, 10% of the loan payments go towards interest?

Theorem 1.1 (Loan Payment Formula, Installment Loans)**.**

$$PMT = \frac{P \times \left(\frac{APR}{n}\right)}{\left[1 - \left(1 + \frac{APR}{n}\right)^{(-nY)}\right]}$$

where

$$\begin{aligned} PMT &= \textit{Regular payment amount} \\ P &= \textit{Starting loan principal (amount borrowed)} \\ APR &= \textit{Annual percentage rate} \\ n &= \textit{Number of payment periods per year} \\ Y &= \textit{Loan term in years} \end{aligned}$$

The interest due each month gradually decreases. The mount paid towards the principal each month gradually increases. Early in the loan term, the portion going toward intense is relatively high and the portion going toward principal is relatively low. As the term proceeds, this pattern gradually reverses.

Ex: For these problems we assume the compounding period is the same as the payment period.

1. Suppose you have student loans totaling \$7500 when you graduate from college. The interest rate is APR=9%, and the loan term is 10 years. What are your monthly payments? How much will you pay over the lifetime of the loan? What is the total interest you will pay on the loan.

   Solution. Using the formula we have monthly payments of

   $$\begin{aligned} PMT &= \frac{\$7500 \times \left(\frac{0.09}{12}\right)}{\left[1 - \left(1 + \frac{0.09}{12}\right)^{(-12(10))}\right]} \\ &= \frac{\$7500 \times (0.0075)}{1 - (1.0075)^{-120}} \\ &= \$95.01. \end{aligned}$$

   Since we pay this amount monthly for 10 years we have

   $$\frac{\$95.01}{\text{month}} \times \frac{12 \text{ month}}{\text{year}} \times 10 \text{ years} = \$11,401.20.$$

   Our loan amount is \$7500, therefore we paid \$11,401.20-\$7500=\$3901.20 in interest. □

2. For the loan in example 1, calculate the portions of your payment that go to the principal and to interest during the first three months.

   Solution. Since the loan interest is compounded monthly at a rate of $\frac{0.09}{12} = 0.0075$, every month we pay (loan amount)$\times$0.0075 in interest. Then we take the monthly payment and subtract this number to get the amount going towards the principal. Now subtract the this amount from the principal to get the new loan amount. Then repeat.

End of...	Interest=0.0075× Balance	Payment Toward Principal	New Principal
Month 1	0.0075× $7500=$56.25	$95.01-$56.25=$38.76	$7500-$38.76=$7461.24
Month 2	0.0075× $7461.24=$55.96	$95.01-$55.96=$39.05	$7461.24-$39.05=$7422.19
Month 3	0.0075× $7422.19=$55.67	$95.01-$55.67=$39.34	$7422.19-$39.34=$7382.85

□

3. You need a $6000 loan to buy a used car. Your bank offers a 3-year loan at 8%, a 4-year loan at 9% and a 5-year loan at 10%. Calculate your monthly payments and total interest over the loan term with each option.

Solution. First lets do the 3 year loan.

$$PMT = \frac{\$6000 \times \left(\frac{0.08}{12}\right)}{\left[1 - \left(1 + \frac{0.08}{12}\right)^{(-12(3))}\right]}$$
$$= \$188.02.$$

Over a three year period, you will pay 36 months$\times\frac{\$188.02}{\text{month}}$=$6768.72. Therefore you will pay $6768.72-$6000=$768.72 in interest.
Next lets do the 4 year loan.

$$PMT = \frac{\$6000 \times \left(\frac{0.09}{12}\right)}{\left[1 - \left(1 + \frac{0.09}{12}\right)^{(-12(4))}\right]}$$
$$= \$149.31.$$

Over a four year period, you will pay 48 months$\times\frac{\$149.31}{\text{month}}$=$7166.88. Therefore you will pay $7166.88-$6000=$1166.88 in interest.
Finally lets do the 5 year loan.

$$PMT = \frac{\$6000 \times \left(\frac{0.1}{12}\right)}{\left[1 - \left(1 + \frac{0.1}{12}\right)^{(-12(5))}\right]}$$
$$= \$127.48.$$

Over a five year period, you will pay 60 months$\times\frac{\$127.48}{\text{month}}$=$7648.80. Therefore you will pay $7648.80-$6000=$1648.80 in interest.

□

4. Suppose you have a credit card balance of $2300 with an annual interest rate of 21%. You decide to pay off you balance over 1 year. How much will you need to pay each month? Assume you make no further credit card purchases.

Solution. Every month you will need to pay

$$PMT = \frac{\$2300 \times \left(\frac{0.21}{12}\right)}{\left[1 - \left(1 + \frac{0.21}{12}\right)^{(-12(1))}\right]}$$
$$= \$214.16.$$

□

5. Paul has gotten into credit card trouble. He has a balance of \$9500 and just lost his job. His credit card company charges interest of APR=21%, compounded daily. Suppose the credit card company allows him to suspend his payments until he finds a new job but continues to charge interest. If it takes him a year to find a new job, how much will he owe when he starts his new job?

   Solution. From a previous section, we know we can calculate interest compounded n times in a year by the given formula,

$$\begin{aligned} A &= P\left(1+\frac{APR}{n}\right)^{nY} \\ &= \$9500 \times \left(1+\frac{0.21}{365}\right)^{365(1)} \\ &= \$11,719.23. \end{aligned}$$

   He will have to pay over \$2200 in additional interest if he can't find a job for a year. □

1.2 Mortgage

Definition 1.3: A home mortgage is an installment loan designed specifically to finance a home
Definition 1.4: A down payment is the amount of money you must pay up front in order to be given a mortgage or other loan.
Definition 1.5: Closing costs are fees you must pay in order to be given the loan. They may include a variety of direct costs, or fee charged as points, where each point is 1% of the loan amount. In most cases, lenders are required to give you a clear assessment of closing costs before you sign for the loan.
Definition 1.6: The simplest type of home loan is a fixed rate mortgage, in which you are guaranteed that the interest rate will not change over the life of the loan.

Ex:

1. You need a loan of \$100,000 to buy your new home. The bank offers a choice of a 30-year fixed rate loan at an APR of 5% or a 15-year fixed rate loan at 4.5%. Compare your Monthly payments and total loan cost under the two options. Assume that the closing costs are the same in both cases and therefore do not affect the choice.

   Solution. Lets first do the 30-year loan at an APR of 5%. Notice we have

$$\begin{aligned} PMT &= \frac{\$100000 \times \left(\frac{0.05}{12}\right)}{\left[1-\left(1+\frac{0.05}{12}\right)^{(-12(30))}\right]} \\ &= \$536.82. \end{aligned}$$

   Over a thirty year period, you will pay 360 months$\times\frac{\$536.82}{\text{month}}\approx$\$193,255. Therefore you will pay \$193,255-\$100000=\$93,255 in interest.
   Next lets do the 15 year loan.

$$\begin{aligned} PMT &= \frac{\$100000 \times \left(\frac{0.045}{12}\right)}{\left[1-\left(1+\frac{0.045}{12}\right)^{(-12(15))}\right]} \\ &= \$764.99. \end{aligned}$$

   Over a 15 year period, you will pay 180 months$\times\frac{\$764.99}{\text{month}}\approx$\$137,698. Therefore you will pay \$137,698-\$100000=\$37,698 in interest.

You will pay about \$230 a month more for the 15 year loan, but you will dance almost \$56,000 in interest. □

2. Lets do a more realistic loan here in Hawaii. The average condo price in Hawaii is \$400,000. Lets say you have a \$100,000 down payment, so you only need a loan for \$300,000. The bank offers a 30-year fixed rate loan at an APR of 5%. What are your monthly payments and total loan cost?

   Solution. Similarly as above, we have

$$\begin{aligned} PMT &= \frac{\$300000 \times \left(\frac{0.05}{12}\right)}{\left[1 - \left(1 + \frac{0.05}{12}\right)^{(-12(30))}\right]} \\ &= \$1,610.46. \end{aligned}$$

   Over a thirty year period, you will pay 360 months$\times\frac{\$1610.46}{\text{month}}\approx$\$579,765.60. Therefore you will pay \$579,765.60-\$300000=\$279,765.60 in interest.

   □

3. Great Bank offers a \$100,000, 30 year, 5% fixed rate loan with closing costs of \$500 plus 1 point. Big Bank offers a lower rate of 4.75% on a 30 year loan, but with closing costs of \$1000 plus 2 points. Evaluate the two options.

   Solution. For the 30 year loan with an fixed interest of 5%, we would have monthly payments of

$$\begin{aligned} PMT &= \frac{\$100000 \times \left(\frac{0.05}{12}\right)}{\left[1 - \left(1 + \frac{0.05}{12}\right)^{(-12(30))}\right]} \\ &= \$536.82, \end{aligned}$$

   while the amount you pay monthly for the 4.75% we have

$$\begin{aligned} PMT &= \frac{\$100000 \times \left(\frac{0.0475}{12}\right)}{\left[1 - \left(1 + \frac{0.0475}{12}\right)^{(-12(30))}\right]} \\ &= \$521.65. \end{aligned}$$

   The difference is about \$15 per month. However the closing cost is $\$500 + \$100000(0.01) = \$1500$ for the 5% interest rate loan and $\$1000 + \$100000(0.02) = \$3000$. Meaning we have to pay an additional \$1500 dollars up front. Notice since we save \$15 dollars a month on payments it would take

$$\frac{\$1500}{\frac{\$15}{\text{month}}} = 100 \text{ month} = 8 \text{ and} \frac{1}{3} \text{ year}.$$

   This being said we would eventually save money. For almost 22 years we would be pocketing the savings of \$15 a month. □

4. An alternative strategy to the mortgage option in EX (1) is to take the 30 year loan at 5%, but try to pay it off in 15 years by making larger payments than are required. To carry our this plan, how much would you have to pay each month? Discuss the pros and cons of this strategy.

Solution. To pay of the loan in 15 years we will need to pay monthly,

$$PMT = \frac{\$100000 \times \left(\frac{0.05}{12}\right)}{\left[1 - \left(1 + \frac{0.05}{12}\right)^{(-12(15))}\right]}$$
$$= \$790.79.$$

We said previously if we want to pay the loan off in 30 years we need to pay about \$537 a month. We will be paying \$254 a month more. Also previously we showed that if we had an interest rate at 4.5% APR and paid it off in 15 years we would pay \$764.99 a month. This is a difference of \$26 dollars. This says that we will be paying more a month than the standard 15 year loan, and we will also be paying more interest at 5% APR instead of 4.5% APR. The biggest reason to do this is the flexibility in the monthly payments. Thought it would cost more money to pay back the loan in 15 years we have the option not to. We can range out monthly payments to as little as \$537 a month. □

5. You have a choice between a 30 year fixed rate loan at 4% and an ARM=(Adjustable Rate Mortgage) with a first year rate of 3%. Neglecting compounding and changes in principal, estimate your monthly savings with the ARM during the first year on a \$100,000 loan. Suppose that the ARM rate rises to 5% by the third year. How will your monthly payments be affected.

AMR or adjustable rate mortgages usually guarantee to maintain their low APR for the first 6 months to a year. After this however the APR can change. Most AMR also include a rate can that cannot be exceeded.

Solution. Since we are neglecting the compounding interest and changes in principal, we only care about the difference in the interest. For the 30 year fixed, we have, $\$100000 \times 4\% = \$4000$. With the AMR we would pay interest of $\$100000 \times 3\% = \$3000$. Over a period of a year we would end up paying about $\frac{\$1000}{12} \approx \$83$ a month more with a fixed rate mortgage. However after 3 years the percentage switches where the AMR is 1% higher than the fixed rate, so now we will pay about \$83 more with the AMR loan. Remember this is assuming that we are neglecting the compounding of interest and changes in principal. This is a very crude estimate.

□